

# TWELVE NEW PRIMITIVE BINARY TRINOMIALS

RICHARD P. BRENT AND PAUL ZIMMERMANN

**ABSTRACT.** We exhibit twelve new primitive trinomials over  $\text{GF}(2)$  of record degrees 42 643 801, 43 112 609, and 74 207 281. In addition we report the first Mersenne exponent not ruled out by Swan's theorem [10] — namely 57 885 161 — for which no primitive trinomial exists. This completes the search for the currently known Mersenne prime exponents.

Primitive trinomials  $x^r + x^s + 1$  of degree  $r \leq 32\,582\,657$  were reported in [5]. We have completed a search for all larger Mersenne prime exponents found by the GIMPS project [7]. Twelve new primitive trinomials were found (see Table 1).

$r$	$s$	Date
42 643 801	55981, 3706066, 3896488, 12899278, 20150445	2009
43 112 609	3569337, 4463337, 17212521, 21078848	2009
57 885 161	none	2013
74 207 281	9156813, 9999621, 30684570	2016

TABLE 1. New primitive trinomials  $x^r + x^s + 1$  of degree a Mersenne exponent  $r$ , for  $s \leq r/2$ . For smaller exponents, see references in [5] or our web site [1].

Our search used the algorithm of [4], relying on fast arithmetic in  $\text{GF}(2)[x]$ ; details are given in [2]. For the squaring of polynomials over  $\text{GF}(2)[x]$ , we used (since 2016) the new `_pdep_u64` Intel intrinsic, which gave a speedup of a factor about 2.5 over the algorithm described in [3, §4]. On a 3.3Ghz Intel Core i5-4590, together with improvements in the `gf2x` library, we were able to square a degree-74 207 280 polynomial in about 2 milliseconds, and to multiply two such polynomials in about 700 milliseconds. As in [5], we produced certificates for non-primitive trinomials (a certificate is simply an encoding of a nontrivial factor of smallest degree). The certificates were checked independently with Magma and NTL. A 3.3Ghz Intel Core i5-4590 takes only 22 minutes to check the certificates of all 37 103 637 reducible trinomials ( $s \leq r/2$ ) of degree  $r = 74\,207\,281$  with our `check-nt1` program based on NTL [9], the largest factor having degree 19 865 299 for  $s = 9\,788\,851$ .

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AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, AUSTRALIA  
*E-mail address:* `trinomials@rpbrent.com`

INRIA NANCY - GRAND EST, VILLERS-LÈS-NANCY, FRANCE  
*E-mail address:* `Paul.Zimmermann@inria.fr`